3 - Reduction of singularities

mercoledì 8 novembre 2023 10:00

Embedded resolution of planar curves

Let $f \in \mathbb{C}[[x,y]]$ be a formal power series (with $f \neq 0$)

Write $f = f_m + f_{m+1} + \dots$ with f_j homogeneous polynomials of degree j, and $f_m \neq 0$. The number $m \in \mathbb{N}$ is called the multiplicity of f at 0.

We will assume that $m \geq 1$, so that f defines a (formal) singular curve $C = \{f = 0\} \subset (\mathbb{C}^2, 0)$ at 0.

Often $f \in \mathbb{C}\{x,y\}$, i.e., $f = \sum_{i,j} f_{ij} x^i y^j$, with $|f_{ij}| \leq \rho^{i+j}$ for some $\rho > 0$

Notice that C is regular at $0 \iff m = 1$.

We want to eliminate the singularities of C via blow-ups.

Def: let $\pi : X_\pi \to (\mathbb{C}^2, 0)$ be the blow-up of 0. The strict transform of C by π is defined as $C_\pi = \overline{\pi^{-1}(C \setminus \{0\})}$.

Algebraically: π is described in two charts:

U_x: $\pi(x_1, y_1) = (x_1, x_1 y_1) \rightsquigarrow \pi^* f = \sum_{j \geq m} x_1^j f_j(1, y_1) = x^m \underbrace{\left(f_m(1,y_1) + x f_{m+1}(1,y_1) + \dots \right)}_{f_\pi}$

At any point $p_1 = (0, a_1)$ in the exceptional divisor, we get a formal power series $(f_\pi)_{p_1}$ defining locally C_π.

Theorem (embedded resolution of planar curves). Let C be a reduced planar curve. Then there exists a finite sequence π of blow-ups (of singular points of the strict transforms of C) such that C_π is a regular curve.

Local Computations

When we blow-up the origin via π, $\forall p \in \pi^{-1}(0)$ we have that $m_p(f_\pi) \leq m_0(f)$, with equality if and only if $f_m(x,y) = c_1 (y - a_1 x)^m$.

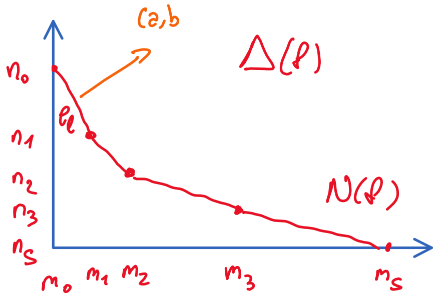
Or $f_m(x,y) = c x^m$, but we can avoid this situation by exchanging x and y.

By proceeding by induction on $m = m_0(f)$, we may assume we are in this case.
Up to a linear change of coordinates $y^{(1)} = \sqrt[m]{c_1}(y - a_1 x)$, we may assume that $f_m(x,y) = y^m$.
We blow-up the origin again: $m_p(f_{\pi_2}) \leq m$, with equality if and only if $f_{\pi_1,m}(x,y) = (y - a_2 x)^m$. We may assume $a_2 = 0$ by a change of coordinates $y^{(2)} = y^{(1)} - a_2 x^2$.
By induction we get that $m_p(f_{\pi_k}) = \dots = m_p(f_0) = m \iff$ up to a change of coordinates of the form $y \rightsquigarrow cy - a_1 x - \dots - a_k x^k$, we get that $f_{\pi_k}(x_k, y_k) = y_k^m + \text{h.o.t.}$.
In other terms, in these new coordinates, $f = y^m + \sum_{i,j} f_{ij} x^i y^j$ ($\leftarrow i + kj > km$)
If this happens indefinitely, we have that $f = (y - \phi(x))^m$ $\phi \in \mathbb{C}[[x]]$
But then C is not reduced, a contradiction. □

Newton's rotating rulers

Theorem. Let $C = \{f(x,y) = 0\}$ be a curve passing through 0. Suppose $x \nmid f$.
Then $\exists \phi \in \mathbb{C}[[x^{\frac{1}{m}}]]$ for some $m \in \mathbb{N}^*$ so that $f(x, \phi) = 0$.
($\leftarrow$ Puiseux series)

Write $f = \sum f_{ij} x^i y^j$. By assumption, $n := \min\{j \mid f_{0j} \neq 0\} < +\infty$. (|| $V_{x,0}(f)$ local intersection with $\{x=0\}$)
On $\mathbb{R}^2_{\geq 0}$, consider $\Delta(f) = \text{ConvHull}\left(\{(i,j) + \mathbb{R}^2_{\geq 0} \mid f_{ij} \neq 0\}\right)$, and $N(f) = \partial\Delta(f)$, called the Newton polygon of f.
Set (m_j, n_j) the points in $N(f)$ where $N(f)$ is not C^1, and $m_j \nearrow$, $n_j \searrow$.
We may assume $m_s = 0$ (if not, $\phi = 0$)
$(0, n_0)$ is called leading vertex v_e
$(0, n_0) \overset{e_e}{—} (m_1, n_1)$ is called leading edge
Take $(a,b) =: \omega$ to be a primitive ($\gcd(a,b) = 1$) vector in $(\mathbb{N}^*)^2$ orthogonal to e_e



The initial part of f (w.r.t. ω) is $in_\omega(f) = \sum_{(i,j)\in e_\ell} f_{ij} x^i y^j$

We now perform the ω-weighted blow-up $\pi : X_\pi \to (\mathbb{C}^2, 0)$

We work on the chart: $(x_1, y_1) \overset{\pi}{\mapsto} (x_1^a, x_1^b y_1)$

$$\pi^* f = \sum f_{ij} x_1^{ai+bj} y_1^j = x_1^{bn} \Big(\underbrace{\sum_{(i,j)\in e_\ell} f_{ij} y_1^j}_{P(y_1)} + \langle x \rangle \Big) \qquad x_1^{bn} P(y_1) = \pi^* in_\omega(f)$$

The strict transform $C_\pi = \pi^* C$ intersects the exceptional divisor $E_1 = \{x_1 = 0\}$ at the roots of P.

A) If $\forall \alpha$ root of P, $ord_P(\alpha) < n$, we have that $\nu_{x_1,\alpha}(f_\pi)$ strictly decreases, and we conclude by recursion: we translate: $(\tilde{x}_1, \tilde{y}_1) \mapsto (\tilde{x}_1, \tilde{y}_1 + \alpha)$ $\quad \left(f_\pi = \frac{\pi^* f}{x_1^{bn}} \right)$

$\phi = \alpha x^{b/a} + x^{b/a} y_1$ for the Puiseux series.

B) If $P = c(y_1 - \alpha)^n$, we must have $\alpha \neq 0$ (e_ℓ contains at least two vertices), and (by Newton binomial formula), $(\frac{b}{a}, n-1) \in e_\ell$, and $a = 1$

In this case, we perform the change of coordinates $(x', y') \mapsto (x', y' + \alpha x'^b)$

$\phi = \alpha x^b + y'$ for the Puiseux series

This change of coordinates erases the vertices in e_ℓ (but for the leading vertex v_ℓ), and we get a new leading edge e'_ℓ associated to a weight $\omega' = (a', b')$ with $\frac{b'}{a'} > \frac{b}{a}$.

This process either finishes, or we fall ∞-many times in the situation B.

In this case, the formal change of coordinates $(x^\infty, y^\infty) \mapsto (x^\infty, y^\infty + \alpha_1 x^{b_1} + \alpha_2 x^{b_2} + \dots)$ gives a map $f(x^\infty, y^\infty)$ with $(y^\infty)^n \mid f$, and C is not reduced. $\square$

Rem: For the Puiseux series, there are choices on the root α, that correspond to two different phenomena:

- different branches of C
- different lifts of a branch of C on orbifold charts.

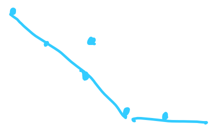
Ex: $y^3 - 3xy^2 + 3x^2y + x^3 + x^2y^2 - x^4 = (y-x)^3 + x^2(y-x)(y+x)$

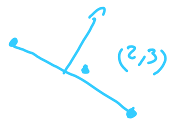
$\rightsquigarrow (y')^3 + y'x'^2(y'+2x')$

$y'^2 + x'^2y' + 2x'^3$

$x_2^6 y_2^2 + x_2^7 y_2 + 2x_2^6$

$x_2^6\left((y_2^2+2) + x_2y_2\right)$





Splits into 2 $\quad y_2 = y_2' \pm i\sqrt{2} \quad y_2'(y_2' \pm 2i\sqrt{2}) + x_2(y_2' \pm i\sqrt{2}) = 0$

Maximal contact

We have seen how we are led to often change coordinates polynomially in order to have them better adapted to branches: $y=0$ should have maximal contact (= local intersection) with the branch that we want to either solve or parametrise (Puiseux).

Regular curves that maximise the intersection multiplicity with a given branch are called "curves of maximal contact", and are a geometric tool to get a cleaner exposition of the resolution algorithm.

In this setting, they always exist (convergent)

Rem: the analogous concept in higher dimensions is a fundamental tool for the proof of Hironaka's theorem.

Reduction of Singularities of Foliations

Thm (Seidenberg, 1968) Let $\mathcal{F}$ be a foliation on $(\mathbb{C}^2,0)$.
There exist a finite sequence of blow-ups of (singular points) $\pi: X_\pi \to (\mathbb{C}^2,0)$ so that each singularity is elementary. Up to further blow-up, we can obtain reduced singularities.

Def: a foliation $\mathcal{F}$ in $(\mathbb{C}^2,0)$ is elementary if it is generated by X that is either regular ($X(0)\neq 0$) or whose linear part $X^{(1)}$ is non nilpotent.

In the latter case, denote by $\lambda_1\neq 0$, λ_2 the eigenvalues of $X^{(1)}$, and set $\lambda := \frac{\lambda_2}{\lambda_1} \in \mathbb{C}$.

Then X is reduced if X is regular, or $\lambda \in \mathbb{C}\setminus\mathbb{Q}_{>0}$.

Rem. In the terminology of [McQuillan-Panazzolo]:

elementary $\Leftrightarrow$ log-canonical

reduced $\Leftrightarrow$ canonical

elementary non-reduced $\Leftrightarrow$ radial

Local computations

The approach via local computations consists in studying the behavior of singularities under blow-up.

We describe a foliation $\mathcal{F}$ via a tangent 1-form $\omega = f\,dx + g\,dy$.

In order to control the singularities under blow-up, we introduce two quantities.

$\nu_0(\mathcal{F}) = \nu_0(\omega) = \min(\mathrm{ord}_0(f), \mathrm{ord}_0(g))$ the order;

$\mu_0(\mathcal{F}) = \mu_0(\omega) = (f=0)\cdot(g=0)_0$ the multiplicity, or Milnor number

($\uparrow$ intersection multiplicity at 0)

These quantities do not depend on the choice of ω, nor on the coordinates (x,y) at 0 (we call ν_0 and μ_0 "invariants").

Rem: For an hypersurface singularity $X = \{s=0\}$, the Milnor number is defined as $\mu_0(X) = \dfrac{\mathbb{C}[[x^1,\ldots,x^d]]}{\langle \frac{\partial s}{\partial x^1},\ldots,\frac{\partial s}{\partial x^d}\rangle} = \left(\frac{\partial s}{\partial x_1}=0\right)\cdot\ldots\cdot\left(\frac{\partial s}{\partial x^d}=0\right)_0$

Hence $\mu_0(\omega) = \mu_0(X)$ if $\omega = ds$.

Notice that $\nu_0(\mathcal{F}) = 1 \iff \mathcal{F}$ is elementary or nilpotent

$\mu_0(\mathcal{F}) = 1 \iff \mathcal{F}$ is elementary not Saddle-Node (with Π over $\mathcal{F}$)

If we blow-up $0 \in \mathbb{C}^2$, we can express the projection $\pi : X_\pi \to (\mathbb{C}^2, 0)$ via two charts: on U_x we have $\pi(x_1, y_1) = (x_1, x_1 y_1)$, and in U_y we get $(x_1 y_1, y_1)$.

By lifting ω, we get $\pi^* \omega = f(x_1, x_1 y_1)\, dx_1 + g(x_1, x_1 y_1)(y_1 dx_1 + x_1 dy_1)$

$$= \underbrace{(xf + yg)}_{h}\circ\pi \cdot \frac{dx_1}{x_1} + (xg)\circ\pi\, dy_1$$

If $\nu = \nu_0(\omega)$, then

$$\pi^*\omega = x_1^{\nu}\Big(\underbrace{f^{(\nu)}(1, y_1) + y_1 g^{(\nu)}(1, y_1)}_{r(y_1) \,=\, h^{(\nu+1)}(1, y_1)} + \langle x_1 \rangle\Big) dx_1 + x_1^{\nu+1}\Big(g^{(\nu)}(1, y_1) + \langle x_1 \rangle\Big) dy_1.$$

We distinguish two cases:

- non-dicritical case: $r(y_1) \not\equiv 0$. The saturation is $\omega_\pi = x_1^{-\nu}\pi^*\omega$, and $p_0 = (0, y_0) \in S(\omega_\pi)$ is singular $\iff y_0$ is a root of r.

Rem: $\infty \in S(\omega_\pi) \iff h^{(\nu+1)}([0:1]) = 0 \iff h^{(\nu+1)}$ has not the term $y^{\nu+1} \iff x \mid h^{(\nu+1)} \iff x \mid g^{(\nu)}$

Hence ω_π has at most $\nu+1$ singularities (and exactly $\nu+1$ if counted with mult.).

- dicritical case: $r(y_1) \equiv 0$. Notice that $g^{(\nu)} \not\equiv 0$ (or we would have $g^{(\nu)} \equiv f^{(\nu)} \equiv 0$ and $\nu_0(\omega) > \nu$). Hence $\omega_\pi = x_1^{-(\nu+1)}\pi^*\omega$ defines a generically transverse foliation $\mathcal{F}_\pi$ with finitely many ($\leq \nu$) points $p \in E_0$ where $g^{(\nu)}(p) = 0$, that are either tangency or singular points of $\mathcal{F}_\pi$. ($T(\omega_\pi)$ = tangency, $S(\omega_\pi)$ = singular, $TS(\omega_\pi) = T \cup S$).

In fact, $p_0 = (0, y_0) \in S(\omega_\pi) \iff g^{(\nu)}(p_0) = 0$ and $h^{(\nu+2)}(p_0) = 0$.

Rem: $\infty \in TS(\omega_\pi) \iff f^{(\nu)}$ has no term $x^{\nu} \iff y \mid f^{(\nu)}$.

$\infty \in S(\omega_\pi) \iff x \mid f^{(\nu)}$ and $x \mid h^{(\nu+2)}$, or equivalently $x \mid g^{(\nu+1)}$.

Ex: $\omega = y\,dx - (x + \varepsilon y)\,dy$ $\quad h = \varepsilon y^2$:
- $\varepsilon = 0$ dicritic $\omega_\pi = -dy$, one singularity
- $\varepsilon \neq 0$ non dicritic $\omega_\pi = \varepsilon y^2 dx - x(1 + \varepsilon y)\, dy$

Prop. Let $C_f = \{f=0\}$ and $C_g=\{g=0\}$ be curves at $0\in\mathbb{C}^2$ without common branches. Then $\mu_0(f,g) = \nu_0(f)\cdot\nu_0(g) + \sum_{p\in E_0}\mu_p(f_\pi, g_\pi)$ where $(C_f)_\pi = \{f_\pi=0\}$ is the strict transform of C_f, and similarly for C_g.

Proof: $C_f\cdot C_g \underset{\uparrow}{=} \pi^* C_f\cdot\pi^* C_g = (\tilde C_f + \nu_f E)\cdot(\tilde C_g+\nu_g E) =$
projection formula

$$= \tilde C_f\cdot\tilde C_g + \nu_f \underbrace{E\cdot\tilde C_g}_{-E\cdot\nu_g E} + \nu_g \underbrace{E\cdot\tilde C_f}_{-E\cdot\nu_f E} + \nu_f\nu_g \underbrace{E\cdot E}_{-1}$$

$$= \sum_{p\in E}\mu_p(\tilde f,\tilde g) + \nu_f\nu_g.$$ $\square$

We apply this computation to PFAFFIAN forms (= 1-forms).

error in the proof (dicritical case)

Prop ([IY Thm 8.33]) Let $\mathcal{F}$ be a singular foliation in $(\mathbb{C}^2,0)$, and π the blow-up of 0.
We have: $\sum_p \mu_p(\mathcal{F}_\pi) = \mu_0(\mathcal{F}) + k + 1 - k^2$ (☆) where

$$k = \mathrm{ord}_E(\pi^*\mathcal{F}) = \mathrm{ord}_E(\pi^*\omega) = \begin{cases}\nu_0(\omega) & \text{non dicritical case}\\ \nu_0(\omega)+1 & \text{dicritical case}\end{cases}$$

Proof: Up to a linear change of coordinates, we may assume that $\infty\notin S(\omega_\pi)$

- Non-dicritical case: this is equivalent to $x\nmid g^{(\nu)}$, i.e. $(x\cdot g)_0=\nu$ $(=k)$
- Dicritical case: this is equivalent to $x\nmid f^{(\nu)}$ or $x\nmid g^{(\nu+1)}$.

Notice that from $y\, g^{(\nu)} = -x f^{(\nu)}$, we get $x \mid g^{(\nu)}$.

If $x\nmid g^{(\nu+1)}$, then $(x\cdot g)_0 = \nu+1$ $(=k)$.

If $x\mid g^{(\nu+1)}$, we have $x\nmid f^{(\nu)}$. Up to a change of coordinates $(x,y) \overset{\Phi}{=} (x+\tilde y^2, \tilde y)$, we get again $x\nmid g^{(\nu+1)}$.

Write $f^{(\nu)} = y\ell$ $\quad g^{(\nu)} = -x\ell$ $\quad \ell$ homogeneous of degree $\nu-1$, with $\ell = y^{\nu-1}+\dots$

Then $\Phi^*\omega = (\tilde y\,\ell\circ\Phi + f^{>\nu}\circ\Phi)(dx + 2\tilde y\, d\tilde y) - (x+\tilde y^2)\,\ell\circ\Phi\, d\tilde y$

$$= (\tilde y\,\ell\circ\Phi + f^{>\nu}\circ\Phi)\,dx + (-x\,\ell\circ\Phi + \tilde y^2\,\ell\circ\Phi + 2\tilde y\, f^{>\nu}\circ\Phi)\,d\tilde y$$

$$(\tilde y\,\ell(x,\tilde y) + \mathcal{M}^{\nu+1})\,dx + (-x\,\ell(x,\tilde y) + \tilde y^{\nu+1} + x\,\underset{\uparrow}{p^{(\nu)}}(x,y) + \mathcal{M}^{\nu+2})\,dy$$

homogeneous of degree ν

Notice that in particular, $x \nmid g$, which is equivalent to asking that $\{g=0\}$ and $\{h=0\}$ have no common branches. In fact:
If ϕ is irreducible and $\phi | g$, $\phi | h$, then $\phi | h - yg = xf$. If $\phi | f$, then ω is not reduced $\Rightarrow \phi | x$. Hence a common branch must be $\{x=0\}$.

Write $\omega = f\,dx + g\,dy$, $\pi(x_1, y_1) = (x_1, x_1 y_1)$, so that $\pi^*\omega = \frac{h\circ\pi}{x_1} dx_1 + xg\circ\pi\, dy_1$
We compute $(h \cdot g)_0 = (xf \cdot g)_0 = (x \cdot g)_0 + (f \cdot g)_0 = k + \mu_0$
Moreover, $\omega_\pi = \frac{\pi^*\omega}{x^k} = \frac{h\circ\pi}{x^{k+1}} dx + \frac{g\circ\pi}{x^{k-1}} dy$

Denote by $D_g = \{g=0\}$ and $D_h = \{h=0\}$ the divisors associated to g and h.
Then $\sum \mu_p(\omega_\pi) = (\pi^* D_h - (k+1)E)(\pi^* D_g - (k-1)E)$

$$= \underbrace{\pi^* D_h \cdot \pi^* D_g}_{D_h \cdot D_g = \mu_0 + k} - (k+1)\underbrace{E \cdot \pi^* D_g}_{0} - (k-1)\underbrace{\pi^* D_h \cdot E}_{0} + (k^2-1)\underbrace{E^2}_{-1}$$

(proj. formula)

□

Proof of reduction to elementary singularities.
Let $\mathcal{F}$ be a non-elementary singularity, of order ν and multiplicity μ.
By ☆, we have that $\mu_p(\omega_\pi) \leq \sum \mu_p(\omega_p) = \mu_0 + \begin{cases} -\nu^2 + \nu + 1 & \text{non dicritical} \\ -\nu^2 - \nu + 1 & \text{dicritical} \end{cases}$

Notice that $-\nu^2 + \nu + 1 < 0 \ \forall \nu \geq 2$, and $-\nu^2 - \nu + 1 < 0 \ \forall \nu \geq 1$.
We deduce that the multiplicity strictly decreases under blow-up, unless $\mathcal{F}$ is non-dicritical of order $1 = \nu$.
The only non-elementary case to be studied is $\mathcal{F}$ nilpotent ($\Rightarrow$ non-dicritical)
$\omega = f(x,y)\,dx + \underbrace{(y + \tilde{g}(x,y))}_{g}\,dy$, $\nu_0(f), \nu_0(\tilde{g}) \geq 2$. Set $\mu = \mu_0(\omega)$.
(coeff 1 up to ~)

We may assume $g = y$: by Weierstrass preparation, we can write $g = (y + g_0(x))u$ with u a unit; notice that $\mathrm{ord}_x g_0 = \mathrm{ord}_x(g(x,0))$, but in general $g_0 \not\equiv g(\cdot, 0)$.

We then change coordinates $(x,y') = (x, y+g_0(x))$ $y = y'-g_0(x)$, and get

$$\omega = f(x, y'-g_0(x))\,dx + y'\,u(x, y'-g_0(x))\left(dy' - \frac{dg_0}{dx}(x)dx\right)$$
$$= f'(x,y')\,dx + y'u'(x,y')\,dy'$$

We can finally divide by the unity u', and get an equivalent 1-form $f''(x,y')dx + y'dy'$

In this case, we write $f'' = \underbrace{a(x)}_{\sum_{j\geq\mu} a_j'' x^j} + \underbrace{y\tilde{f}(x,y)}_{\sum_{i,j,\, j\geq 1} f_{ij} x^i y^j}$, with $\mathrm{ord}_0(a) = \mu$.

If we denote by double prime the coeff. in the new form, we get $f_{11}'' = \dfrac{f_{11} - g_{20}}{b}$

We blow-up, and get $\omega_1 = \left(\dfrac{f(x,xy)}{x} + y^2\right)dx + xy\,dy$

$$\left(\frac{a(x)''}{x} + y\tilde{f}(x,xy) + y^2\right)dx + xy\,dy$$
$$\left(a_\mu x^{\mu-1} + \langle x^\mu\rangle + f_{1,1}xy + \langle xy^2, x^2\rangle + y^2\right)$$

We get a unique singularity at $p_1 = [1:0]$, with the following invariants.

If $\mu \geq 3$: $(\nu_1 = 2, \mu_1 = \mu+1)$ case A

If $\mu = 2$: $(\nu_1 = 1; \mu_1 = 3)$ case B

Case A: we blow-up further; we get $h^{(3)} = x^3 a_3 + f_{11}x^2y + 2xy^2 \neq 0$ (non dicritical).

$\Rightarrow$ we have a singularity at ∞ (of multiplicity 1), and at least another.

Then we get $\sum_{p\in S_2} \mu_p(\omega_2) = \mu+1-1 = \mu$, and $\#S_2 \geq 2$: the multiplicity drops.

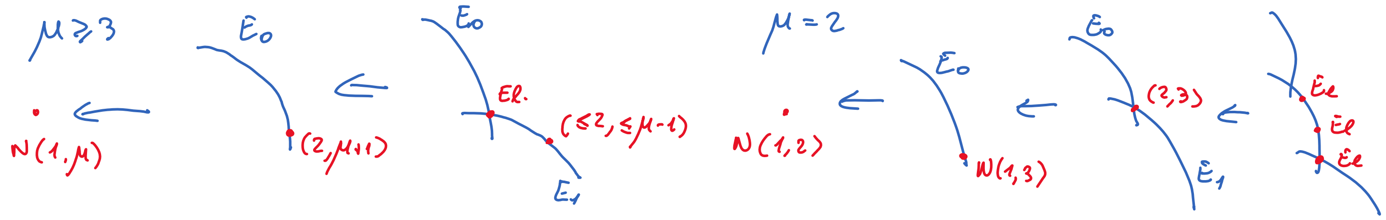
Case B: the singularity of p_1 is nilpotent:

$$\omega_1 = \left(a_2 x + a_3 x^2 + f_{11}xy + y^2 + xy\langle x,y\rangle\right)dx + xy\,dy$$

This singularity has multiplicity 3, and the new f_{11}^1 is $1-1=0$

We deduce that $h_2^{(3)} = l_0 x^3 + l_2 xy^2$ $l_0, l_2 \neq 0$ which has three distinct zeroes. Hence after 3 blow-ups, we got from $\overset{\nu\ \mu}{(1,2)}$ to $3\times(1,1)$
↑ elementary

Nilpotent singularities:



Newton polygon approach

[Pelletier, Panazzolo]

In this approach, we work with the vector field $\chi = f\partial_x + g\partial_y$ instead.

While in general vector fields do not pull back, they do as long as we blow-up a singularity: If $\pi(x_1,y_1)=(x_1, x_1y_1)$, alors $\pi^{-1}(x,y) = \left(x, \frac{y}{x}\right)$, et

$$\pi^*\chi_{p_1} = \left((\pi^{-1})_*\chi\right)_{p_1} = d\pi^{-1}_p \chi_p = f\circ\pi\, \partial_{x_1} + \frac{-y\, f\circ\pi + x\, g\circ\pi}{x^2}\partial_{y_1} =$$

$$d\pi^{-1}_p = \begin{pmatrix} 1 & 0 \\ -\frac{y}{x^2} & \frac{1}{x}\end{pmatrix} \qquad = f\circ\pi\,\partial_{x_1} + \frac{-y_1 f\circ\pi + g\circ\pi}{x_1}\partial_{y_1}$$

But $x_1 \mid -y_1 f\circ\pi + g\circ\pi \Leftrightarrow f(0)=g(0)=0 \Leftrightarrow 0\in S(\chi)$.

Rem: Dans les calculs, il vaut mieux laisser $d\pi^{-1}$ en fonction de x_1, y_1.

$$d\pi = \begin{pmatrix} 1 & 0 \\ y_1 & x_1\end{pmatrix} \quad d\pi^{-1} = \begin{pmatrix} 1 & 0 \\ -\frac{y_1}{x_1} & \frac{1}{x_1}\end{pmatrix} \qquad \pi^*\chi = f\circ\pi\underbrace{\left(\partial_{x_1} - \frac{y_1}{x_1}\partial_{y_1}\right)}_{d\pi^{-1}(\partial_x)} + g\circ\pi\underbrace{\left(\frac{1}{x_1}\partial_{y_1}\right)}_{d\pi^{-1}(\partial_y)}$$

More generally, consider weighted blow-ups: $\pi(x_1,y_1) = (x_1^a ; x_1^b y_1)$,

or toric blow-ups: $\pi(x_1,y_1) = (x_1^a y_1^c, x_1^b y_1^d)$ $\quad \pi(z) = z^A \qquad A = \begin{pmatrix} a & c \\ b & d\end{pmatrix}$

$$d\pi = \begin{pmatrix} a x_1^{a-1} y_1^c & c x_1^a y_1^{c-1} \\ b x_1^{b-1} y_1^d & d x_1^b y_1^{d-1}\end{pmatrix} \qquad \det d\pi = (ad-bc)\, x_1^{a+b-1} y_1^{c+d-1}$$

$$(d\pi)^{-1}_{(x,y)} = \frac{1}{\det A}\begin{pmatrix} \frac{d x_1}{x_1^a y_1^c} & \frac{-c x_1}{x_1^b y_1^d} \\ \frac{-b y_1}{x_1^a y_1^c} & \frac{a y_1}{x_1^b y_1^d}\end{pmatrix} = \frac{1}{\det A}\begin{pmatrix} x_1 & 0 \\ 0 & y_1\end{pmatrix}\begin{pmatrix} d & -c \\ -b & a\end{pmatrix}\begin{pmatrix} \frac{1}{x} & 0 \\ 0 & \frac{1}{y}\end{pmatrix}$$

$$= \begin{pmatrix} x_1 & 0 \\ 0 & y_1\end{pmatrix} A^{-1}\begin{pmatrix} \frac{1}{x} & 0 \\ 0 & \frac{1}{y}\end{pmatrix}$$

In particular, on the logarithmic basis $(x\partial_x, y\partial_y)$, π^* acts by sending

$$\underbrace{\lambda z^I z\partial_z}_{\lambda=(\alpha,\beta)} := x^i y^j \big(\underbrace{\alpha}_{f_{ij}} x\partial_x + \underbrace{\beta}_{g_{ij}} y\partial_y\big) \text{ to } x_1^{ai+bj} y_1^{ci+dj} \cdot \frac{1}{\det A}\big((\alpha d-\beta c)x_1\partial_{x_1} + (-\alpha b+\beta a)y_1\partial_{y_1}\big) = A^{-1}\lambda \cdot z_1^{IA} z_1\partial_{z_1}$$

Hence we write $\chi = F x\partial_x + G y\partial_y$ with $F = \sum_{i\geq -1,\, j\geq 0} F_{ij} x^i y^j$ ($= f_{i+1,j}$), $G = \sum_{i\geq 0,\, j\geq -1} G_{ij} x^i y^j$ ($= g_{i,j+1}$)

$= \sum \Lambda_I z^I \cdot z\partial_z$, notation $\Lambda_I = (F_I, G_I)$

Rem: $\Lambda_{(-1,j)} = (F_{-1,j}, 0)$ $\Lambda_{(i,-1)} = (0, G_{i,-1})$

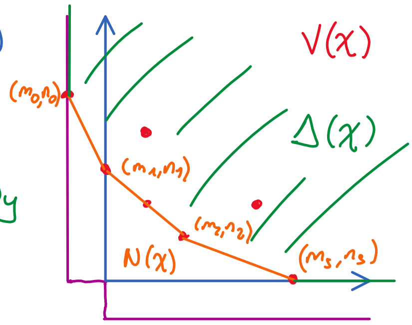
Def The Newton polygon of $\chi = F x\partial_x + G y\partial_y$ w.r.t. $z=(x,y)$:

$V^z(\chi) := \{ I \mid \Lambda_I \neq 0\}$, $\Delta^z(\chi) = \mathrm{CvxHull}(V(\chi) + \mathbb{R}^2_{\geq 0}) \subseteq \{(-1,0),(0,-1)\} + \mathbb{R}^2_{\geq 0}$

Newton polygon: $N^z(\chi)$: union of bounded faces of $\Delta(\chi)$

$(m_0,n_0)\ldots(m_s,n_s)$ vertices of $N(\chi)$ $(n_\ell \searrow, m_\ell \nearrow)$

Ex: $\chi = (y^5 - x^2y^4 + x^3y + x^6)\partial_x + (y^4 - 2xy^3 + x^2y^2 - 2x^4y^3)\partial_y$



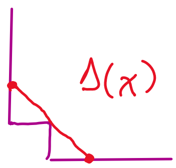
Prop: 1) χ is regular $\iff \lambda_{(-1,0)} \neq 0$ or $\lambda_{(0,-1)} \neq 0 \iff (-1,0)$ or $(0,-1) \in N(\chi)$

$\Rightarrow (0,0) \in \Delta(\chi) \setminus N(\chi)$

2) χ is elementary non-regular $\iff (0,0) \in N^{Lz}(\chi)$ $\forall L \in GL(2,\mathbb{C})$

$\chi^{(1)} = (f_{00}x + f_{-1,1}y)\partial_x + (g_{1,-1}x + g_{00}y)\partial_y$

If $\lambda_{0,0} = (0,0)$, $\chi^{(1)}$ non-nilpotent $\iff f_{-1,1}, g_{1,-1} \neq 0$



If $f_{00} + g_{00} \neq 0 \Rightarrow \mathrm{tr}\,\chi^{(1)} \neq 0 \Rightarrow$ Non nilpotent.

If $f_{00} + g_{00} = 0$: nilpotent $\iff \det \chi^{(1)} = 0$: might happen $\chi^{(1)} = \begin{pmatrix} a & b \\ -\frac{a^2}{b} & -a \end{pmatrix}$ ($a \neq 0$, $b \neq 0$)

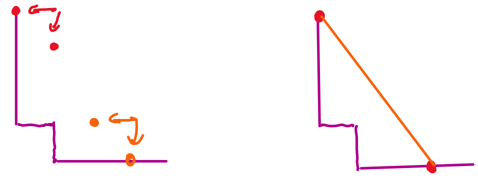
We can avoid this up to linear change of coords

3) χ is saturated $\Rightarrow m_0 \leq 0,\ n_s \leq 0$:

$x \nmid \chi \Rightarrow$ there must be a monomial $y^j\partial_x$ $((-1,j))$ or $y^j\partial_y$ $((0,j-1))$

$y \nmid \chi \Rightarrow$ there must be a monomial $x^i\partial_x$ $((i-1,0))$ or $x^i\partial_y$ $((i,-1))$

Eg: $\chi = y^3\partial_x + x^2\partial_y$



Eg: $\chi = x^3\partial_x + y^2\partial_y$

Eg: $\chi = (x^4 - y^3)\partial_x + xy\partial_y$

4) χ is tangent to $x=0$ $\iff$ there are no monomials $\underset{(-1,j)}{y^j\partial_{x_j}}$ $\iff$ $m_0 = 0$.

Let now perform a weighted blow-up of weights $\omega = (a,b)$, and see how the vector field is transformed $\quad A = \begin{pmatrix} a & 0 \\ b & 1 \end{pmatrix} \quad A^{-1} = \begin{pmatrix} \frac{1}{a} & 0 \\ -\frac{b}{a} & 1 \end{pmatrix}$

$$\to \pi^* \Lambda_I z^I z\partial_z = A^{-1}\Lambda_I z_1^{IA} z_1\partial_{z_1} = x_1^{ia+jb} y_1^j \frac{1}{a}\Big(F_I x_1\partial_{x_1} + \underbrace{(-bF_I + aG_I)}_{H_I} y_1\partial_{y_1}\Big)$$

Set $H := -bF + aG$. We denote by $F^{(c)} = \sum_{\langle I,\omega\rangle = c} F_I z^I$ the ω-homogeneous part of F of degree c (analogously for G, H).

C_0: ω-order of χ $= \min\{c \mid \underbrace{F^{(c)} x\partial x + G^{(c)} y\partial_y}_{\text{for } c = c_0:\ \omega\text{-initial part}} \neq 0\}$

Then we have: $\chi = \sum_{c \geq c_0} F^{(c)} x\partial x + G^{(c)} y\partial_y$.

$$\begin{aligned} a\pi^*\chi &= F\circ\pi\, x_1\partial_{x_1} + H\circ\pi\, y_1\partial_{y_1} \\ &= \sum_{c\geq c_0} x_1^c \left(F^{(c)}(1,y_1)\, x_1\partial_{x_1} + H^{(c)}(1,y_1)\, y_1\partial_{y_1}\right) \\ &= x_1^{c_0} H^{(c_0)}(1,y_1) y_1\partial_{y_1} + x_1^{c_0+1}\left(F^{(c_0)}(1,y_1)\,\partial_{x_1} + H^{(c_0+1)}(1,y_1) y_1\partial_{y_1}\right) + x_1^{c_0+2}\eta_2 \end{aligned}$$

As we did for the case of 1-forms, we distinguish 2 cases:

(A) • (ω-) NON-DICRITICAL CASE: $H^{(c_0)} \not\equiv 0$. $\chi_\pi = x_1^{-c_0}\pi^*\chi = H^{(c_0)}(1,y_1) y_1\partial_{y_1} + x_1\eta_1$

χ_π is tangent to $E_\omega = \{x_1 = 0\}$

Singularities of: zeroes of $H^{(c_0)}(1,y_1)\cdot y_1 =: R(y_1)$.

$I_\omega = (m_\omega, n_\omega)$, (m'_ω, n'_ω) the extremes of $N(\chi) \cap \{\langle I,\omega\rangle = c_0\} =: N_\omega(\chi)$

Similarly, $(m_H, n_H), (m_H', n_H')$ are the extremes of $N_\omega(H)$.

By construction: $\deg R = 1 + n_H \leq 1 + n_\omega$; $\operatorname{ord}_0 R = 1 + n_H' \geq 1 + n_\omega'$

Notice that if $m_\omega = -1$, then $F_{I_\omega} \neq 0$, $G_{I_\omega} = 0 \Rightarrow H_{I_\omega} = -b F_{I_\omega} \neq 0$, and $n_H = n_\omega$.

Notice that any $\alpha \in E_\omega \setminus \{\infty\}$ root of $y_1 H$ of multiplicity e_α gives a singularity of χ_π with $(0, e-1) \in \Delta(\chi_\pi)$. Since χ_π is tangent to E_ω, we get $(\tilde{m}_0, \tilde{n}_0) = (0, e_\alpha - 1)$. In particular, we have decreased the value of the leading vertex, unless $y_1 H \simeq (y_1 - \alpha)^{n_0+1}$

Studying ∞ corresponds to studying 0 in the other chart.

We deduce that $e_\infty = m_H + 1$. In general this number can be greater than $n_0 + 1$.

But for this to happen, we must have $bF_I = aG_I$ $\forall I \in N_\omega(\chi) \setminus N_\omega(H)$, $i < m_H$.

In particular the order of $G^{(C_0)}_{(x_1,1)}$ at 0 is m_ω, and χ_π has a monomial $x_1^{m_\omega} y_1 \partial y_1 \rightsquigarrow (m_\omega, 0) \in V(\chi_\omega)$. (and $(i,0) \notin V(\chi_\omega)$ $\forall i < m_\omega$)

Being χ_π tangent to $E_\omega = \{y_1 = 0\}$, we get $(m_\omega, 0) \in N(\chi_\omega)$

(B) • (ω)- DICRITICAL CASE : $H^{(C_0)} \equiv 0$. In this case $F^{(C_0)} \simeq G^{(C_0)}$, and $N_\omega(F) = N_\omega(G) = N_\omega(\chi) \subseteq \mathbb{R}^2_{\geq 0}$.

$$\chi_\pi = x_1^{-C_0-1} \pi^* \chi = \underbrace{F^{(C_0)}(1, y_1)}_{\text{polynomial of degree } n_\omega} \partial x_1 + H^{(C_0+1)}(1, y_1) y_1 \partial y_1 + x_1 \eta_2$$

at a root $\alpha \in E_\omega \setminus \{\infty\}$ of order e, we get the monomial $y_1^e \partial x_1 \rightsquigarrow (-1, e-1)$, and the leading exponent always drops.

At ∞, the computation is analogous, with the roles of F and G (x and y) interchanged. The multiplicity is less than m_ω

Resolution algorithm

We define the leading edge by (m_0, n_0) (leading height: n_0) $(m_0 \leq 0)$, and the leading edge e_ℓ as the first edge on the left.

We take ω as the positive primitive vector orthogonal to e_ℓ.

We blow-up with weight ω.

At ∞, the height becomes 0 or 1 (i.e., elementary or nilpotent), because $m_0 \le 0$ (we also exchange the role of the coordinates, i.e., work with the coordinates (y_1, x_1), where $\pi(x_1, y_1) = (x_1^a y_1, y_1^b)$).

At $E_\omega \setminus \{\infty\}$, the leading height strictly decreases unless $\mathcal{X}$ is ω-non-dicritical and $y_1 H^{(c_0)}(1, y_1) \asymp (y_1 - \alpha)^{n_0+1}$

If $\alpha = 0 \Rightarrow F_{m_1,n_1} \neq 0 \Rightarrow y_1^{n_1} x_1 \partial_{x_1}$ is a monomial in $\mathcal{X}_\pi$ that gives $(0, n_1) \in N(\mathcal{X}_\pi)$, with $n_1 < n_0$

If $\alpha \neq 0 \Rightarrow a = 1$, we change coordinates $y' = y - \alpha x$

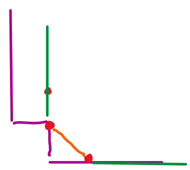
We get the phenomenon analogous to the case of curves.

By iterating the process, we get singularities with $n_0 \le 0$, which are elementary

Remarks

1) A priori we could have formal curves of maximal contact.

Ex: Euler vector field: $\mathcal{X} = x^2 \partial_x + (y - x)\partial_y$.



Notice that this singularity is already elementary

We change coordinates $x' = x$, $y' = y - \alpha x^n$ $\Phi_n : (x', y') \mapsto (x', y' + \alpha x'^n)$

$d\Phi_{n,\alpha}^{-1} = \begin{pmatrix} 1 & 0 \\ -n\alpha x^{n-1} & 1 \end{pmatrix}$ $\mathcal{X}_1 = \Phi_{1,1}^* \mathcal{X} = x^2(\partial_x - \partial_y) + y\partial_y$

$\mathcal{X}_2 = \Phi_{2,1}^* \mathcal{X}_1 = x^2(\partial_x - 2x\partial_y) + y\partial_y$

$\Phi_{3,2}^* \mathcal{X}_2 = x^2(\partial_x - 6x^2\partial_y) + y\partial_y$

$\vdots$

$y_\infty = y - \sum_{n=1}^{\infty} (n-1)!\, x^n$ ← divergent. $\mathcal{X} = x^2 \partial_x - y^\infty \partial_{y_\infty}$

If we get a similar phenomenon for $n_0 \ge 1$, we would get $N^{(x,y^\infty)}(\mathcal{X}) \subseteq \{j \ge 1\}$, and $\mathcal{X}^\infty$ is not saturated, $\{y_\infty = 0\}$ is a curve of singularities. But $\mathrm{Sing}(\mathcal{X})$ is an analytic subspace, hence the maximal contact curve must converge

2) The algorithms in literature are slightly different in the choice of the leading vertex and edge, see [PELLETIER], or [PANAZZOLO]).

3) This approach has been followed by PANAZZOLO (see also [McQUILLAN-PANAZ.]) to reduce singularities of vector fields in dimension 3.

Notice that starting from dimension 3, we <u>need</u> weighted blow-ups.

Ex (PANAZZOLO) Does not reduce by simple blow-ups (of loci singular for χ_π)

$$\chi = x(x\partial_x - \alpha y\partial_y - \beta z\partial_z) + xz\partial_y + (y-\lambda x)\partial_z \qquad \alpha,\beta\geq 0,\ \lambda>0$$

$Sing(\chi) = \{x=y=0\}$. We can either blow-up 0 or $L=\{x=y=0\}$

CASE Bl_0: $\pi=(x_1, x_1y_1, x_1z_1)$.

$$\chi = x\cdot x\partial_x + x\left(\frac{z}{y}-\alpha\right)y\partial_y + \left(\frac{y-\lambda x}{z} - \beta x\right)z\partial_z$$

$$\pi^*\chi = x\cdot x\partial_x + x\left(\frac{z}{y}-\alpha-1\right)y\partial_y + \left(\frac{y-\lambda}{z} - (\beta+1)x\right)z\partial_z$$

$Sing\ \pi^*\chi = \{x = y-\lambda = 0\}$. We translate coordinates at $p=(0,\lambda,\lambda(\alpha+1))$, and get

$$\pi^*\chi = x\cdot x\partial_x + x(z-(\alpha+1)y)\partial_y + \big(y-(\beta+1)x(z+\lambda(\alpha+1))\big)\partial_z \ :$$

Same singularity with coeff $(\alpha+1, \beta+1, \lambda(\alpha+1)(\beta+1))$

CASE Bl_L: $\pi(x_1,y_1,z_1) = (x_1, x_1y_1, z_1)$

$$\pi^*\chi = x\cdot x\partial_x + x\left(\frac{z}{xy}-\alpha-1\right)y\partial_y + \left(\frac{y-\lambda}{z}-\beta\right)xz\partial_z$$

$Sing\ \pi^*\chi = \{x = z - \cancel{(\alpha+1)}xy = 0\}$. We translate at $(0,\lambda,0)$, and get

$$\pi^*\chi = x\cdot x\partial_x + (z-(\alpha+1)x(y+\lambda))\partial_y + \left(\frac{y}{z}-\beta\right)xz\partial_z$$

$$= x\cdot x\partial_x + x\left(\frac{y}{z}-\beta\right)z\partial_z + \left(\frac{z-(\alpha+1)\lambda x}{y} - (\alpha+1)x\right)y\partial_y\ ,$$

which, with respect to coords (x,z,y),

is again of the same form with parameters $(\beta, \alpha+1, \lambda(\alpha+1))$.

4) Reduction of singularities hold for codim. 1 foliations in any dimension, by [CANO-CERVEAU, 1992] (non-dicritical) and [CANO, 2004].